Field theoretic approach to Gaussian-Heisenberg crossover behavior

1976 J. Phys. A: Math. Gen. 9 961

(http://iopscience.iop.org/0305-4470/9/6/015)

J. Phys. A: Math. Gen., Vol. 9, No. 6. 1976. Printed in Great Britain.

Field theoretic approach to Gaussian–Heisenberg crossover behaviour

I D Lawrie

School of Mathematical and Physical Sciences, University of Sussex, Falmer, Brighton, England

Received 28 November 1975, in final form 11 February 1976

Abstract. The renormalization prescription of t'Hooft and Veltman is used to construct crossover scaling functions for the susceptibility and free energy in an isotropic n vector model. Some difficulties in interpreting the ϵ expansion in this context are discussed, and the formalism is illustrated by a calculation of the expansion factor α^2 of a polymer in dilute solution.

1. Introduction

An aspect of the renormalization group approach to critical phenomena which is currently receiving much attention is the problem of the crossover behaviour of thermodynamic functions between regions in which they exhibit the simple kinds of behaviour generated by different fixed points of the renormalization group transformation. In this article, we study a simple crossover problem, namely the dependence of the correlation functions in ϕ^4 theory on the scaling variable $z \sim g_0 \tau^{-\epsilon/2}$, where g_0 is the four-spin coupling constant, τ the reduced temperature and $d = 4 - \epsilon$ the dimensionality of space. For large z, the theory is dominated by the critical behaviour generated by the Wilson–Fisher fixed point, and the susceptibility, for example, diverges as

$$\chi \sim \tau^{-\gamma} \tag{1.1}$$

where γ is the well known critical exponent. When z is small, the tricritical behaviour, governed by the mean-field exponents emerges, and one finds that

$$\chi \sim \tau^{-1}. \tag{1.2}$$

The essence of the problem is to discover a scaling function, $X(z)$, such that

$$\chi = \tau^{-1} X(z) \tag{1.3}$$

reproduces the simple forms (1.1) and (1.2) in the appropriate limits, and correctly describes the susceptibility for general values of z. $\epsilon/2$ is the crossover exponent for this problem (Riedel and Wegner 1969, 1970).

Brézin *et al* (1973a) have examined the critical region in considerable detail, using the Callan–Symanzik equations in renormalized perturbation theory, and the present investigation uses a related renormalization procedure, due to t'Hooft and Veltman (1972) (see also t'Hooft 1973, Collins and Macfarlane 1974), which offers several useful features. The renormalization group equation resulting from this procedure is

homogeneous in the correlation function (i.e. it does not involve the correlation function with an extra ϕ^2 insertion) at all momenta, and the mass parameter is multiplicatively renormalized, which has the consequence that both the bare and the renormalized masses are proportional to the reduced temperature, in the case of a magnetic system. Thus the critical value, m_{0c}^2, of the mass parameter is zero, independently of the value of g_0. The two independent parameters in the Lagrangian (2.1) can therefore be taken to be $\tau = m_0^2$ and the dimensionless quantity z, and dimensional analysis now shows that (1.3) is the only possible form for the susceptibility. This somewhat unusual feature appears to stem from the fact that certain divergent contributions to the Feynman integrals are neglected in the dimensional regularization procedure. If a finite momentum cut-off were to be used, an additive renormalization would also be required. In fact, a homogeneous renormalization group equation due to Weinberg (1973) has already been used (Zinn-Justin 1973), but the present technique seems to be calculationally simpler.

As this renormalization procedure has not, to the author's knowledge, been used previously in the study of critical phenomena, we give in § 2 a summary of the method, and indicate how the critical exponents and scaling laws can be obtained directly from the definitions of the thermodynamic functions. In § 3 we calculate, to order ϵ, the crossover scaling functions for the free energy and the susceptibility, and it will be seen that the extension to higher orders is a matter only of labour. We note, however, that there are difficulties in interpreting the ϵ expansion result outside the critical region. In § 4, we illustrate the formalism by calculating the expansion factor α^2 of a polymer in dilute solution. This problem has recently been studied by Burch and Moore (1976) who have calculated polymer properties using the essentially phenomenological approach of Riedel and Wegner (1974) to crossover problems. We have not obtained a closer fit to the data than theirs, but we find a marked improvement between the first two orders of the ϵ expansion.

2. Renormalization group equation

We consider the traditional model of an n component scalar field theory defined, in terms of unrenormalized quantities, by the Lagrangian density:

$$-\mathscr{L}(x) = \frac{1}{2}(\nabla\phi_0(x))^2 + \frac{1}{2}m_0^2\phi_0^2(x) + \frac{1}{4!}g_0(\phi_0^2)^2 \tag{2.1}$$

where ϕ_0^2 stands for $\Sigma_{\alpha=1}^n \phi_0^\alpha \phi_0^\alpha$. The renormalization procedure involves defining the following renormalized quantities:

$$\phi(x) = Z_3^{-1/2}\phi_0(x)$$
$$m^2 = Z_m^{-1}m_0^2$$
$$u = \mu^{-\epsilon}Z_1^{-1}Z_3^2 g_0.$$

Here μ is an arbitrary parameter, with the dimensions of mass, which means that the renormalized coupling constant u is dimensionless. It is possible to show that the renormalization constants Z_1, Z_3, Z_m are given by the procedure we shall describe as functions of u alone. In fact they are power series in u, whose first terms are unity, and they are infinite in four dimensions.

In terms of the renormalized quantities, the Lagrangian reads

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 \tag{2.2}$$

$$-\mathscr{L}_0 = \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2$$

$$-\mathscr{L}_1 = \frac{1}{4!}\mu^\epsilon u Z_1(u)(\phi^2)^2 + \frac{1}{2}(Z_3(u)-1)(\nabla\phi)^2 + \frac{1}{2}(Z_3(u)Z_m(u)-1)m^2\phi^2$$

and the connected correlations functions, $\tilde{G}^{(s)}(x_1, \ldots, x_s)$, defined by

$$\frac{\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\ldots\delta_{\alpha_{s-1}\alpha_s} + \text{permutations}}{1.3.5\ldots(s-1)}\tilde{G}^{(s)}(x_1\ldots x_s) = \langle\phi^{\alpha_1}(x_1)\ldots\phi^{\alpha_s}(x_s)\rangle_c$$

are calculated in perturbation theory with the Feynman rules obtained from (2.2) in the usual way. The one-particle-irreducible vertex functions, obtained from the $G^{(s)}$ by deleting all graphs which can be disconnected by cutting a single line, and removing from those which remain the propagators associated with external legs, are denoted, in momentum space, by $\Gamma^{(s)}(p_1\ldots p_s; u, m, \mu)$. By convention, the leading term of $\Gamma^{(2)}(p^2)$ is $-(p^2+m^2)$, which makes $\Gamma^{(2)}$ equal to minus the inverse propagator:

$$\Gamma^{(2)}(p^2) = -[G^{(2)}(p^2)]^{-1}. \tag{2.3}$$

The crux of the renormalization prescription is to define Z_1, Z_2 and Z_m so as to make the three quantities

$$\Gamma^{(2)}(p^2=0), \qquad \frac{\partial}{\partial p^2}\bigg|_{p^2=0}\Gamma^{(2)}(p^2) \qquad \text{and} \qquad \Gamma^{(4)}(p_i=0)$$

finite in four dimensions, to all orders in perturbation theory. Using the technique of dimensional regularization, we can cast the Feynman integrals into a form in which their divergences appear as poles at $\epsilon = 0$, and the prescription offered by t'Hooft and Veltman is to subtract just these poles, together with their residues at $\epsilon = 0$. Explicitly, figure 1 shows the diagrams contributing to $\Gamma^{(2)}$ to first order in u. The integral A is

$$A = S_d\int k^{d-1}(k^2+m^2)^{-1}\,dk$$

$$= \tfrac{1}{2}S_d(m^2)^{1-\epsilon/2}B(\epsilon/2-1, 2-\epsilon/2) = -S_4 m^2\epsilon^{-1}(1+O(\epsilon))$$

where

$$S_d = \frac{2\pi^{d/2}}{(2\pi)^d\Gamma(d/2)} = (2\pi)^{-d}\times(\text{surface of a } d \text{ dimensional sphere}).$$

Evaluating the vertex function at $p = 0$, we require the first counterterm just to cancel the pole contained in A. That is

$$Z_3 Z_m = 1 + \frac{n+2}{6\epsilon}S_4 u + O(u^2).$$

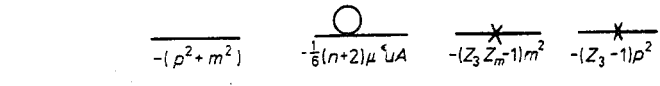

Figure 1. Diagrams contributing to $\Gamma^{(2)}(p^2)$ to order (u).

On differentiating with respect to p^2, since the one-loop diagram is independent of p, the second counterterm is zero to this order, i.e.

$$Z_3 = 1 + O(u^2).$$

A similar condition on $\Gamma^{(4)}(p_i = 0)$ enables us to evaluate Z_1, and the extension of this procedure to higher orders is obvious. The results to order u^2 are:

$$Z_1 = 1 + \frac{n+8}{6\epsilon} S_4 u + \frac{1}{36\epsilon^2}[(n+8)^2 - (5n+22)\epsilon](S_4 u)^2 + O(u^3)$$

$$Z_3 = 1 - \frac{n+2}{144\epsilon}(S_4 u)^2 + O(u^3)$$

$$Z_m = 1 + \frac{n+2}{6\epsilon} S_4 u + \frac{n+2}{36\epsilon^2}\left((n+5) - \frac{5}{4}\epsilon\right)(S_4 u)^2 + O(u^3).$$

The counterterms generated by this procedure are sufficient to define finite correlation functions $G^{(s)}$ and $\Gamma^{(s)}$ for all s. However, we also need to consider matrix elements of the form

$$\langle \phi^2(x_1) \ldots \phi^2(x_r)\phi(y_1) \ldots \phi(y_s) \rangle$$

in which new divergences appear. In order to obtain finite correlation functions, $G^{(r,s)}$, $\Gamma^{(r,s)}$ of this type, we introduce an extra wavefunction renormalization Z_4 and define $\phi^2_{0p}(x) = Z_4 Z_3^{-1} \phi_0^2(x) = Z_4 \phi^2(x)$. Now the unrenormalized vertex functions, $\Gamma_u^{(s)}$, associated with the matrix elements of $\phi_0(x)$ satisfy

$$\Gamma_u^{(s)} = Z_3^{-s/2} \Gamma^{(s)}$$

and

$$\Gamma_u^{(r,s)}(p_i = 0) = \left(2\frac{\partial}{\partial m_0^2}\right)^r \Gamma_u^{(s)}(p_i = 0).$$

Thus

$$\Gamma^{(r,s)}(p_i = 0) = \left(\frac{2Z_4}{Z_3 Z_m}\frac{\partial}{\partial m^2}\right)^r \Gamma^{(s)}(p_i = 0)$$

is finite at $\epsilon = 0$ if $Z_4 = Z_3 Z_m$, which is also found by explicit calculation. The renormalized vertex functions, $\Gamma^{(r,s)}$ are those associated with the matrix elements of ϕ^2_{0p} and ϕ.

The scaling properties of the correlation functions are investigated by use of the renormalization group equation, which is an expression of the fact that the unrenormalized functions are independent of the parameter μ. Making the functional dependences explicit, we have

$$\Gamma^{(r,s)}(p_i; u, m, \mu) = Z_3^{s/2}(u) Z_m^r(u) \Gamma_u^{(r,s)}(p_i; g_0, m_0)$$

and on differentiating with respect to μ, at fixed g_0 and m_0, we get

$$\left[\mu\frac{\partial}{\partial \mu} + \beta(u)\frac{\partial}{\partial u} - \gamma_m(u) m^2 \frac{\partial}{\partial m^2} - \left(\frac{s}{2}\gamma_3(u) + r\gamma_m(u)\right)\right]\Gamma^{(r,s)} = 0 \qquad (2.4)$$

where

$$\beta(u)=\mu\frac{\partial u}{\partial\mu}\bigg|_{g_0m_0}=-\epsilon\left[\frac{\mathrm{d}}{\mathrm{d}u}\ln\left(\frac{uZ_1}{Z_3^2}\right)\right]^{-1}=-\epsilon u\left(1-\frac{n+8}{6\epsilon}S_4u+\frac{3n+14}{12\epsilon}(S_4u)^2+O(u^3)\right)$$

$$\gamma_m(u)=-\frac{\mu}{m^2}\frac{\partial m^2}{\partial\mu}\bigg|_{g_0m_0}=\beta(u)\frac{\mathrm{d}}{\mathrm{d}u}\ln Z_m=-\frac{n+2}{6}S_4u[1-\tfrac{5}{12}S_4u+O(u^2)] \tag{2.5}$$

$$\gamma_3(u)=\mu\frac{\partial}{\partial\mu}\ln Z_3=\beta(u)\frac{\mathrm{d}}{\mathrm{d}u}\ln Z_3=\frac{n+2}{72}(S_4u)^2+O(u^3).$$

Clearly, these functions are closely related to the Callan–Symanzik functions obtained by Brézin *et al* (1973a), and here also, it appears that, at least in perturbation theory, β has a zero of order ϵ, given by

$$S_4u^*=\frac{6}{n+8}\epsilon\left[1+\left(\frac{6}{n+8}\right)^2\frac{3n+14}{12}\epsilon+O(\epsilon^2)\right]$$

and has the general form indicated in figure 2.

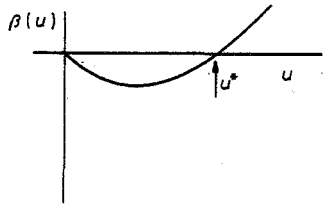

Figure 2. Sketch of $\beta(u)$.

Equation (2.4) may be written more conveniently in terms of the following variables

$$\nu(u)=(2+\gamma_m(u))^{-1}\qquad B(u)=\nu(u)\beta(u)\qquad t=m^2/\mu^2$$

i.e.

$$\left[t\frac{\partial}{\partial t}-\nu(u)\mu\frac{\partial}{\partial\mu}-B(u)\frac{\partial}{\partial u}+\nu(u)\left(\frac{s}{2}\gamma_3(u)+r\gamma_m(u)\right)\right]\Gamma^{(r,s)}=0. \tag{2.6}$$

This equation has the formal solution

$$\Gamma^{(r,s)}(p_i;u,t,\mu)$$

$$=\Gamma^{(r,s)}(p_i;\bar{u}(t),1,\bar{\mu}(t))\exp\left[-\int_1^t\nu(\bar{u}(t'))\left(\frac{s}{2}\gamma_3(\bar{u}(t'))+r\gamma_m(\bar{u}(t'))\right)\frac{\mathrm{d}t'}{t'}\right] \tag{2.7}$$

where the effective coupling constant and mass parameters, $\bar{u}$ and $\bar{\mu}$ are defined by

$$\ln t=\int_u^{\bar{u}(t)}\frac{\mathrm{d}u'}{B(u')} \tag{2.8a}$$

$$\ln\left(\frac{\bar{\mu}(t)}{\mu}\right)=\int_u^{\bar{u}(t)}\frac{\nu(u')\mathrm{d}u'}{B(u')} \tag{2.9a}$$

or equivalently by the differential equations

$$t\frac{\partial}{\partial t}\bar{u} = B(\bar{u}); \qquad \bar{u}(t=1) = u \tag{2.8b}$$

$$t\frac{\partial}{\partial t}\bar{\mu} = \nu(\bar{u})\bar{\mu}; \qquad \bar{\mu}(t=1) = \mu. \tag{2.9b}$$

In the region of interest, $u \leqslant u^*$, $\nu(u)$ is positive, and $B(u)$ has the same general form as $\beta(u)$. Thus (2.8) and (2.9) imply that in the limit $t \to 0$,

$$\bar{u}(t) \approx u^*$$

$$\bar{\mu}(t) \sim t^{\nu}$$

where the critical exponent $\nu = \nu(u^*)$ has exactly the value calculated by other methods.

Since μ is the only dimensional parameter, aside from momenta, it is straightforward to see from (2.7) that the correlation length, ξ, defined by

$$\Gamma^{(2)}(p^2 = -\xi^{-2}; \bar{u}(t), 1, \bar{\mu}(t)) = 0$$

has the form, near $t = 0$,

$$\xi \approx [\bar{\mu}(t)]^{-1} f(u^*) \sim t^{-\nu}.$$

Therefore, if, as is often done, m_0^2 is taken to be linear in temperature, and g_0 to be temperature-independent, the critical point, defined by the divergence of ξ, is at $t = 0$, so t, and therefore m_0^2 are proportional to the reduced temperature, $(T - T_c)/T_c$. Similarly, one can derive from (2.7) the critical behaviour of the susceptibility:

$$\chi^{-1} = \Gamma^{(2)}(p^2 = 0) \sim t^{-\eta\nu}[\bar{\mu}(t)]^2 \sim t^{\gamma}$$

where $\eta = \gamma_3(u^*)$ and $\gamma = (2 - \eta)\nu$ in accordance with the usual scaling laws.

The specific heat is proportional to $\Gamma^{(2,0)}(p_1 = p_2 = 0)$, which is not multiplicatively renormalizable. However, if one assumes that it satisfies (2.7), its behaviour in the critical region is found to be

$$C \sim t^{-\alpha}$$

with $\alpha = 2 - d\nu$, which again agrees with the Widom–Kadanoff scaling laws.

Finally in this section we quote the results, to order ϵ^2 for the critical exponents. For higher order values, the reader is referred to Brézin *et al* (1973b):

$$\nu = \frac{1}{2}\left(1 + \frac{1}{2}\frac{n+2}{n+8}\epsilon + \frac{1}{4}\frac{n+2}{(n+8)^3}(n^2 + 23n + 60)\epsilon^2 + O(\epsilon^3)\right)$$

$$\eta = \frac{1}{2}\frac{n+2}{(n+8)^2}\epsilon^2 + O(\epsilon^3)$$

$$\gamma = 1 + \frac{1}{2}\frac{n+2}{n+8}\epsilon + \frac{1}{4}\frac{n+2}{(n+8)^3}(n^2 + 22n + 52)\epsilon^2 + O(\epsilon^3).$$

3. Crossover scaling functions

In the last section, we obtained the well known ϵ expansion expressions for the critical exponents from the zero-mass behaviour of the renormalized correlation functions,

which are expressed in terms of the renormalized parameters, u, t and μ. However, we are now interested in the behaviour of the theory in terms of m_0 and g_0, which define the model, for general values of these parameters, and the physical quantities we wish ultimately to calculate are related to the matrix elements of the unrenormalized field ϕ_0. In the renormalized theory, as long as u (or $\bar{u}$) is of order ϵ, the ϵ expansion coincides with the perturbation expansion in powers of u, which enabled us to calculate the critical exponents as power series in ϵ. In the perturbation expansion of the renormalization constants, however, the coefficient of u^s is of order $(1/\epsilon)^s$, so that the transition back to the unrenormalized theory is not quite straightforward, if we still wish to use the ϵ expansion.

In order to make this transition, we go back to equations (2.5) which are the analogue in our procedure of the differential recursion relations used in previous investigations (Riedel and Wegner 1974, Nelson and Rudnick 1975). Figure 2 shows that in the limit $\mu \to \infty$, $u \to 0$, and the Z become equal to unity, as long as u is between 0 and u^* for finite μ, which we assume to be the case. We now write $u = vu^*$, and find that

$$(\beta(u)/u^*) = -\epsilon v(1-v)F(v) \tag{3.1}$$

where $F(v)$ is a power series in which the coefficient of v^s is of order ϵ^s and

$$\epsilon F(1) = \beta'(u^*) = \omega = \epsilon\left(1 - \frac{3(3n+14)}{(n+8)^2}\epsilon + O(\epsilon^2)\right).$$

Now integrating (2.5) we have

$$\ln \mu^{-\epsilon} = \int dv\left(\frac{1}{v} + \frac{\epsilon/\omega}{(1-v)} + O(\epsilon^2 v)\right) \tag{3.2}$$

or

$$\mu^{-\epsilon} = \text{constant}\ v(1-v)^{-\epsilon/\omega}\ e^{O(\epsilon^2 v^2)}$$

and with the boundary condition $\lim_{v\to 0} v\mu^{\epsilon} = g_0/u^*$, we can evaluate the constant of integration to give

$$v(1-v)^{-\epsilon/\omega}\ e^{O(\epsilon^2 v^2)} = \mu^{-\epsilon} g_0/u^* = g$$

or

$$v = \frac{g}{1+g}\left(1 - \frac{\ln(1+g)}{1+g} \cdot 3\frac{(3n+14)}{(n+8)^2}\epsilon + O(\epsilon^2)\right).$$

Thus, by ignoring the fact that $\mu^{\epsilon} u^*$ is itself a function of ϵ, we have found an expression for v in terms of g_0 in the form of an ϵ expansion. Note that g vanishes in the limit $\mu \to \infty$. It is perhaps worth remarking that in (3.2) the term involving the exponent ϵ/ω occurs in exactly the form given, because of the definition $\omega = \beta'(u^*)$. Of course if one wishes to evaluate ω to order ϵ^3 or beyond, the terms denoted $O(\epsilon^2 v)$ must be calculated to the same order.

Equation (2.8) may be integrated in the same manner, with $\bar{u} = \bar{v}u^*$ and $B'(u^*) = \omega\nu$. In this case, we treat $t^{\epsilon/2}$ as if it were independent of ϵ, and obtain

$$t^{-\epsilon/2} = c\bar{v}(1-\bar{v})^{-\epsilon/2\omega\nu}\ e^{O(\epsilon^2 \bar{v}^2)} \tag{3.3}$$

where

$$c = \frac{(1-v)^{\epsilon/2\omega\nu}}{v}\ e^{O(\epsilon^2 v^2)} = g^{-1}\left(1 + \frac{n+2}{2(n+8)}\ln(1+g)\epsilon + O(\epsilon^2)\right).$$

In the limit $\mu \to \infty$, $t = \mu^{-2} m_0^2$, and

$$ct^{\epsilon/2} = u^* m_0^\epsilon / g_0.$$

Equation (3.3) shows explicitly how the theory is dominated by the two fixed points $\bar{v} = 0$ (Gaussian) and $\bar{v} = 1$ (Wilson–Fisher) in the limiting cases of the dimensionless ratio g_0/m_0^ϵ becoming respectively small or large, with the leading corrections to scaling governed by the exponents $\epsilon/2$ or $\omega\nu$ (cf Brézin *et al* 1973a).

$\bar{v}$, and (by integrating (2.9)) $\bar{\mu}$ may now be expressed in the form of an ϵ expansion, as functions of the scaling variable $z = g_0/u^* m_0^\epsilon$ where, again, the ϵ dependence of this variable is ignored. Since we have identified the reduced temperature as m_0^2, we change to the more suggestive notation, $\tau = m_0^2$. The results, in the limit $\mu \to \infty$, are

$$\bar{v}(z) = \frac{z}{1+z}\left(1 - \frac{-n^2+8n+68}{2(n+8)^2}\epsilon \frac{\ln(1+z)}{1+z} + O(\epsilon^2)\right) \tag{3.4}$$

$$\bar{\mu}(\tau, z) = \tau^{1/2}(1+z)^{(1/\epsilon)\{1-2\nu-[(n+2)/4(n+8)^3](-n^2+8n+68)\epsilon^2(1+z)^{-1}+O(\epsilon^3)\}}$$

$$\times \exp\left(\frac{n+2}{4(n+8)^3}(13n+44)\epsilon \frac{z}{1+z} + O(\epsilon^2)\right) \tag{3.5}$$

$$K(z) = \exp\left(\int_1^t \gamma_3(\bar{u}(t'))\nu(\bar{u}(t'))\frac{dt'}{t'}\right) = (1+z)^{-2\eta\nu/\epsilon} \exp\left(\frac{n+2}{2(n+8)^2}\epsilon \frac{z}{1+z} + O(\epsilon^2)\right). \tag{3.6}$$

The susceptibility, $\chi = [\Gamma^{(2)}(p^2 = 0)]^{-1}$ can now be evaluated from (2.7) and is given by

$$\chi = \tau^{-1}(1+z)^{(2/\epsilon)\{\gamma-1+[(n+2)/4(n+8)^3](-n^2+8n+68)\epsilon^2(1+z)^{-1}+O(\epsilon^3)\}}$$

$$\times \left[1 + \frac{n+2}{n+8}\left(\frac{1}{2}(1 + \ln 4\pi - \gamma_E) - \frac{6(n+3)}{(n+8)^2}\right)\epsilon \frac{z}{1+z} + O(\epsilon^2)\right]. \tag{3.7}$$

We can also find an expression for the free energy functional which is, apart from a singular, spin independent term,

$$\Gamma(M, u, t, \mu) = \sum_{s=2}^{\infty} \frac{1}{s!} M^s \Gamma^{(s)}(p_i = 0; u, t, \mu)$$

where $M^\alpha = \langle \phi^\alpha \rangle$ is a constant magnetization. (2.7) may be used to write this in the form

$$\Gamma = \bar{\mu}^d F(y, \bar{u}) \tag{3.8}$$

where $y = M\bar{u}\bar{\mu}^{2-d}K(z)$. Near $\tau = 0$, $y \sim M^2\tau^{-2\beta}$, with $\beta = \frac{1}{2}\nu(d-2+\eta)$, as expected, and the expansion of $F(y, \bar{u})$ in powers of $\bar{u}$ corresponds to the loopwise expansion (Zinn-Justin 1973). To one-loop order, the result is

$$\bar{u}F(y, \bar{u}) = \frac{1}{2}y + \frac{1}{4!}y^2 + \frac{1}{4}S_4\bar{u}\left[(n-1)\left(1+\frac{1}{6}y\right)^2\left[\ln\left(1+\frac{1}{6}y\right) - \frac{1}{2}\right]\right.$$

$$+ \left(1+\frac{1}{2}y\right)^2\left[\ln\left(1+\frac{1}{2}y\right) - \frac{1}{2}\right] + (1 - \gamma_E + \ln 4\pi)(n+2)$$

$$\left.\times \frac{1}{3}y\left(1 + \frac{n+8}{12(n+2)}y\right) + O(\epsilon)\right]. \tag{3.9}$$

At this point, some comments about the form of these results are appropriate. Firstly, we point out that the correlation functions calculated by this method may be

expanded in powers of z, and upon re-expressing them in terms of g_0 and m_0, we recover the expressions obtained from the unrenormalized perturbation expansion, correct to the appropriate order in ϵ. However, the forms given in (3.4)–(3.9) evidently contain more information than the unrenormalized theory, since the coefficient of each power of ϵ is an exact function of z: no assumption has been made about the magnitudes of g_0 and m_0, although by working without a finite momentum cut-off, we have implicitly restricted ourselves to a region in which the lattice spacing or corresponding quantity can properly be neglected. Secondly, the fact that the only 'small' parameter appearing explicitly is ϵ, leads to certain ambiguities of interpretation. When one is calculating only a single number, such as a critical exponent, the situation is clear: whether the ϵ expansion converges or not, one obtains a result which can be compared directly with experiment at each order of approximation. To determine the functional form of the thermodynamic functions in g_0 and m_0, or whatever parameters one starts with, is a somewhat different problem, and in order to make sensible use of the ϵ expansion, one needs some other guide as to what functional form to expect, just as in the critical region one assumes simple scaling behaviour as some power of τ. The scaling variable z seems to arise naturally from the formalism, and one obviously wishes to avoid expanding $\tau^{\epsilon/2}$ in powers of logarithms, but faced with an expression such as (3.7), it is not obvious that one should not, for example, expand in powers of $\ln(1+z)$, keeping only the $O(\epsilon^0)$ term in the exponent. The form given seems to be instructive, in that the simple power law behaviour in the limits of large or small z is readily apparent.

We note, in this connection, that in the limit $\epsilon \to 0$, the well known logarithmic corrections are absent from our result. The reason for this is that the limits $\epsilon \to 0$ and $\mu \to \infty$ do not commute. Thus the form of ϵ expansion we have used is not appropriate for studying the four-dimensional limit of the theory. To lowest order, the susceptibility is

$$\chi = \tau^{-1}[1+(g_0/u^*)\tau^{-\epsilon/2}]^{(n+2)/(n+8)} \tag{3.10}$$

which may be compared with the result of Nelson and Rudnick (1975)

$$\chi = \tau^{-1}[1+(g_0/u^*)(\tau^{-\epsilon/2}-1)]^{(n+2)/(n+8)}. \tag{3.11}$$

This result, in which the logarithmic correction is reproduced, can, in fact, be obtained from the present method by setting $\mu = 1$, and expanding in powers of g_0/u^*. However, since u^* is of order ϵ, this variable is of order ϵ^{-1}, or of order unity, if g_0 is taken to be a multiple of u^*, and such a procedure does not give rise to an ϵ expansion.

The simplest way of illustrating our form for the susceptibility is to evaluate the effective exponent (Riedel and Wegner 1974)

$$\gamma_{\text{eff}}(z) = -\tau\frac{\partial}{\partial\tau}\ln\chi(z)$$

and in figure 3 we draw this quantity to order ϵ and ϵ^2, with $n = d = 3$ and $g_0/u^* = 1$. We note that by setting $g_0/u^* = \frac{1}{2}$ in (3.11), we obtain the same γ_{eff} as from (3.10).

The expressions we have obtained in this section are open to criticism, on the grounds that, while (3.3) manifestly has the correct analytic structure for all t, this does not appear explicitly in the final solutions. Although it is not difficult to obtain from them, by logarithmic differentiation, the corrections to scaling near the Heisenberg fixed point in powers of $\tau^{\omega\nu}$, the scaling variable appropriate to this region is not, strictly speaking, z, but $z^{-2\omega\nu/\epsilon}$. Clearly, it is not possible to exhibit within a single expression

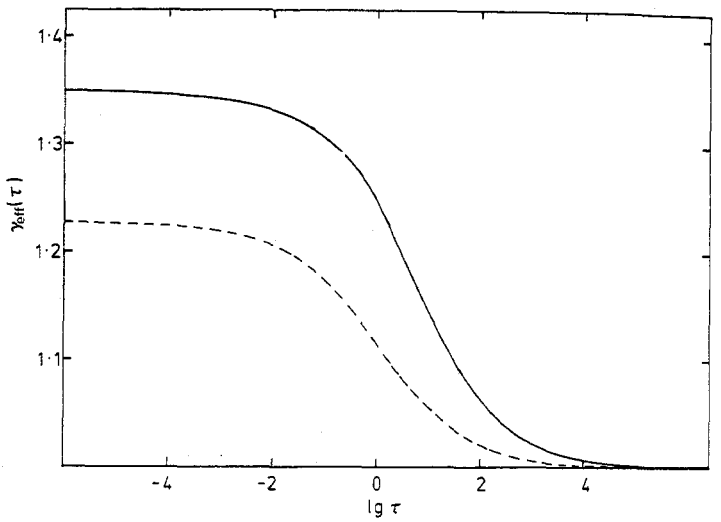

Figure 3. Plot of γ_{eff} with $n = d = 3$ to first (broken curve) and second (full curve) orders in ϵ.

the correct analytic structure near both fixed points, although a parametric representation can achieve this (Bruce and Wallace 1976). Nevertheless, our expressions are correct solutions to (3.3) to $O(\epsilon^2)$, and furthermore, the numerical results of § 4 are not significantly altered by using, for large z, an approximate expression in which the correct analytic form is displayed explicitly. (We are obliged to D J Wallace for this point.)

Another limitation of our method is that it is not straightforward to include non-renormalizable interactions such as ϕ^6, although a calculation along the lines proposed by Wolsky (1974) might be possible, so that the region $g_0 < 0$, which is physically interesting applications, for example to collapsed polymers, and ^{3}He–^{4}He mixtures, is not accessible.

4. Application to polymer chains

We illustrate the use of the formalism developed in the last section, by calculating the expansion factor, α^2, of a polymer in dilute solution. The statistics of a single polymer chain are described (de Gennes 1972, des Cloizeaux 1975, Burch and Moore 1976) by the Lagrangian (2.1) in the limit $n \to 0$. An elegant proof of the connection has been given by Emery (1975). We assume that the field-theoretic parameters are to be identified as follows

$$m_0^2 = 2da^{-2}(1-J)$$
$$g_0 = a^{-\epsilon}u_0[1-(\theta/T)]$$

where a is the length of a flexible unit of the chain, and θ is the Flory temperature, at which the configurations of the chain assume a random walk distribution. The propagator, $\tilde{G}^{(2)}(R)$, is then the generating function of the distributions, $C_N(R)$,

obtained from the inverse Laplace transform

$$C_N(R) = a^{d-2}\frac{1}{2\pi i}\oint \frac{dJ}{J^{N+1}}\tilde{G}^{(2)}(R) \tag{4.1}$$

which represent the number of configurations of a chain of N links, whose end-to-end distance is R. If

$$C_N = a^{-d}\int d^dR C_N(R) = a^{-2}\frac{1}{2\pi i}\oint \frac{dJ}{J^{N+1}}[\Gamma^{(2)}(k^2=0)]^{-1}$$

the mean square end-to-end distance is

$$\langle R^2\rangle = \frac{1}{C_N}2da^{-d}\frac{1}{2\pi i}\oint \frac{dJ}{J^{N+1}}\frac{\partial}{\partial k^2}\bigg|_{k^2=0}[\Gamma^{(2)}(k^2)]^{-1} \tag{4.2}$$

which can be evaluated using the results of the last section:

$$\langle R^2\rangle = Na^2(1+\bar{z})^{(1/4)+(15/128)\epsilon+(17/128)\epsilon(1+\bar{z})^{-1}}\left(1+\tfrac{1}{8}(\ln 4\pi - \tfrac{27}{6})\epsilon\frac{\bar{z}}{1+\bar{z}}\right) \tag{4.3}$$

with

$$\bar{z} = \frac{u_0}{u^*}[1-(\theta/T)]\left(\frac{N}{2d}\right)^{\epsilon/2}$$

We have used the asymptotic expression, valid for large N for the binomial coefficient

$$\binom{-x}{N} \approx (-1)^N\frac{N^{x-1}}{\Gamma(x)}.$$

This result coincides, to lowest order, with that obtained by Stephen (1975). In order to make a comparison with experiment, and with existing theories, we need to identify the variable $\mathscr{Z}$, commonly used in the polymer literature. The expansion of $\alpha^2 = \langle R^2\rangle/Na^2$ in powers of $\mathscr{Z}$, given for example by Yamakawa (1971) is

$$\begin{aligned}\alpha^2 &= 1+\tfrac{4}{3}\mathscr{Z}+O(\mathscr{Z}^2)\\ \mathscr{Z} &= \left(\frac{3}{2\pi a^2}\right)^{3/2}\beta N^{1/2}\end{aligned} \tag{4.4}$$

in three dimensions, where β is the binary cluster integral, and is proportional to $[1-(\theta/T)]$. As anticipated earlier, it is by no means obvious how the ϵ expansion result is to be interpreted. However, the result in perturbation theory, and the generalization of (4.4) to $4-\epsilon$ dimensions are respectively

$$\alpha^2 = 1+\tfrac{1}{3}(4\pi)^{\epsilon/2}S_4u_0[1-(\theta/T)](N/2d)^{\epsilon/2}[\epsilon(1+\tfrac{1}{2}\epsilon)]^{-1}+O(u_0^2) \tag{4.5}$$

and

$$\alpha^2 = 1+(d/2\pi a^2)^{d/2}\beta N^{\epsilon/2}2[\epsilon(1+\tfrac{1}{2}\epsilon)]^{-1}+O(\beta^2). \tag{4.6}$$

Comparison of (4.4)–(4.6) suggests the identification

$$\mathscr{Z} = \frac{1}{6\epsilon}(4\pi)^{\epsilon/2}S_4u^*\bar{z}$$

and inserting this into (4.3) we obtain finally

$$\alpha^2=(1+8\mathscr{Z})^{(1/4)+(15/128)\epsilon+(17/128)\epsilon(1+8\mathscr{Z})^{-1}}\left(1-\frac{3\mathscr{Z}}{1+8\mathscr{Z}}\epsilon\right). \tag{4.7}$$

In the good solvent region, well above θ, that is, for large values of $\mathscr{Z}$, the asymptotic form

$$\langle R^2\rangle \sim N^{2\nu}$$

with $\nu=3/5$, is in good agreement with experiment. This implies that

$$\alpha^2 \sim \mathscr{Z}^{2/5}.$$

Now (4.7) predicts for the exponent of $\mathscr{Z}$ the values 0·25 and 0·37 at zeroth and first orders in ϵ, and we observe, as is often the case, that the first two terms of the ϵ expansion converge towards the experimental data. In figure 4, we give a plot of $\alpha^2(\mathscr{Z})$, and again a slow convergence is apparent, although the actual magnitude of α^2 at large $\mathscr{Z}$ does not agree well with the data.

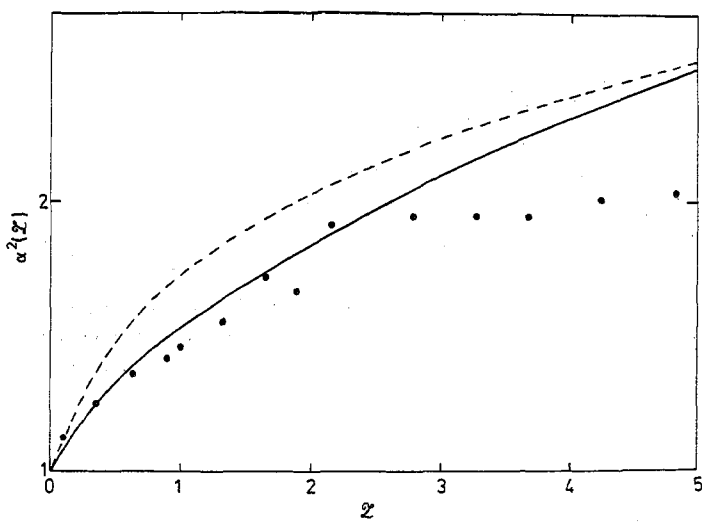

Figure 4. Plot of (4.7) to zeroth (broken curve) and first (full curve) orders in ϵ. Data points are taken from Berry (1966) and are for polystyrene in decalin.

Acknowledgments

The author is indebted to Dr D R T Jones, Dr M A Moore and Dr A Love for many useful conversations, and to the Science Research Council for financial support.

References

Berry G C 1966 *J. Chem. Phys.* **44** 4550
Brézin E, le Guillou J C and Zinn-Justin J 1973a *Phys. Rev.* D **8** 434, 2418
Brézin E, le Guillou J C, Zinn-Justin J and Nickel B G 1973b *Phys. Lett.* **44A** 227

Bruce A D and Wallace D J 1976 *J. Phys. A: Math. Gen.* **9** in the press
Burch D J and Moore M A 1976 *J. Phys. A: Math. Gen.* **9** 435–50
des Cloizeaux J 1975 *J. Physique* **36** 281
Collins J C and Macfarlane A J 1974 *Phys. Rev.* D **10** 1201
Emery V J 1975 *Phys. Rev.* B **11** 239
de Gennes P G 1972 *Phys. Lett.* **38A** 339
t'Hooft G 1973 *Nucl. Phys.* B **61** 455
t'Hooft G and Veltman M 1972 *Nucl. Phys.* B **44** 189
Nelson D R and Rudnick J 1975 *Phys. Rev. Lett.* **35** 178
Riedel E K and Wegner F J 1969 *Z. Phys.* **225** 195
—— 1970 *Phys. Rev. Lett.* **24** 730, 930
—— 1974 *Phys. Rev.* B **9** 294
Stephen M J 1975 *Phys. Lett.* **53A** 363
Weinberg S 1973 *Phys. Rev.* D **8** 3497
Wolsky A M 1974 *Phys. Rev.* B **10** 3877
Yamakawa H 1971 *Modern Theory of Polymer Solutions* (New York: Harper and Row)
Zinn-Justin J 1973 *Lectures delivered at Cargèse Summer Institute* unpublished